Technical Comments

Comment on "Similarity Analysis for Multicomponent, Free Convection"

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Nomenclature

a^* = typical length
c_p, c_p^* = specific capacity at constant pressure, a universal constant for all species
D_i^* = mass transfer coefficient of species i
G = $\rho_{ref}^* U^* a^*/\mu_{ref}^*$, gravity number[1]
g, g^* = magnitude of gravitational acceleration
h = h^*/h_{ref}^*, nondimensional enthalpy
K = $k^*/\rho_{ref}^* c_p^*$, thermometric conductivity (thermal diffusion)
k, k^* = thermal conductivity
L_Q = parabolic partial differential operator involving transport property Q
M = $U^*/(h_{ref}^*)^{1/2}$, compressibility factor[2]
m_i = molecular weight of species i
N = number of species present
Pr = $\mu^* c_p^*/k^*$, Prandtl number
p = $(p^* - p_{ref}^*)/(\mu_{ref}^* U^*/a^*)$, nondimensional pressure above reference
$\bar{p}$ = dimensional pressure over reference
p_e = hydrostatic pressure
$\mathbf{q}$ = $\mathbf{q}^*/U^*$, nondimensional velocity
R = universal gas constant
Sc_i = $\mu^*/\rho^* D_i^*$, Schmidt number for species i
T = temperature
T_B = boiling or sublimation temperature
U^* = $\rho_{ref}^* a^{*2} g^*/\mu_{ref}^*$, quantity with dimensions of a speed
u = velocity component tangential to flat plate
v = velocity component normal to flat plate
x = spatial coordinate tangential to flat plate
$\hat{x}$ = direction in which gravity acts
Y_i = mass fraction of species i
y = spatial coordinate normal to flat plate
$\mathbf{z}$ = $\mathbf{z}^*/a^*$, nondimensional spatial coordinate
β = $-\rho_\infty^{-1}(\partial\rho/\partial T)_\infty$, coefficient of thermal expansion
δ_i = $-\rho_\infty^{-1}(\partial\rho/\partial Y_i)_\infty$, coefficient of composition
ϵ = $\Delta T^*/T_{ref}^* = \Delta h^*/h_{ref}^*$ for c_p^* const
λ_e = $\lambda_{ref}^*/\mu_{ref}$, i.e., $\lambda^*(h_{ref}^*)/\mu^*(h_{ref}^*)$
λ = λ^*/μ_{ref}^*, nondimensional second coefficient of viscosity
$\bar{\lambda}$ = specific heat of vaporization (sublimation)
μ = μ^*/μ_{ref}^*, nondimensional first coefficient of viscosity
ρ = ρ^*/ρ_{ref}^*, nondimensional density
ν^* = μ^*/ρ^*, kinematic viscosity
$\dot{\tau}$ = $\dot{\tau}^*/(\mu_{ref}^* U^*/a^*)$, nondimensional stress tensor
Φ = $\Phi^*/(\mu_{ref}^* U^{*2}/a^{*2})$, nondimensional dissipation

Superscript

$*$ = dimensional quantity

LOWELL and Adams[3] have attempted to use the concepts of similarity to obtain the flow of a mixture of ideal gases along a semi-infinite vertical flat plate under buoyancy. They have adopted Boussinesq-like equations[4] within the framework of boundary-layer theory for the limit of large Grashof number. They have pointed out that multicomponent flow requires 1) the species conservation equations, which they take to be identical in form to the energy equation under Fick's law for diffusion; and 2) a buoyancy force which involves a generalized form for the density above ambient. These observations have been made earlier.[5,6] Unfortunately, Lowell and Adams were unsuccessful in their attempt to generalize the classical similarity solution (which is usually credited to Pohlhausen[7]).

Here we extend Ostrach's derivation[8] of the Boussinesq equations for constant-density ambient conditions, to account for multicomponent flow problems. Also, we point out that in the practically important case of an adiabatically vaporizing (or sublimating) wall, the boundary conditions are compatible with Pohlhausen similarity. This is the free convection analogue of the Emmons problem,[9] which involves only forced convection.

Consider a steady multicomponent flow driven solely by a nondimensionalized temperature difference ϵ impressed across two impermeable walls. The vectors normal to the two walls $\hat{n}_1$ and $\hat{n}_2$ are not both perfectly parallel or antiparallel to the uniform gravitational acceleration. The flowfield is preferably of not too great extent and the time of examination is not indefinitely long. It is convenient to neglect thermal and pressure diffusion, to take the viscosity coefficients as functions of the enthalpy only, to adopt a constant specific heat capacity universal to all species present, and to limit the Schmidt and Prandtl numbers to order-unity constants.

Consider now the special case $\epsilon \ll 1$, and let the mass fraction of each species be a constant for $\epsilon = 0$. The following expansions are adopted (the nondimensionalization is given in the Nomenclature):

$$\rho = 1 + \epsilon\rho_1 + o(\epsilon) \tag{1}$$

$$h = 1 + \epsilon h_1 + o(\epsilon) \tag{2}$$

$$p = p_e(x) + \epsilon p_1 + o(\epsilon) \tag{3}$$

$$\mathbf{q} = 0 + \epsilon\mathbf{q}_1 + o(\epsilon) \tag{4}$$

$$\mu = 1 + \epsilon\mu_1 + o(\epsilon) \qquad \mu_1 = \frac{d\mu(h=1)}{dh} h_1 \tag{5}$$

$$\lambda = \lambda_e + \epsilon\lambda_1 + o(\epsilon) \qquad \lambda_1 = \frac{d\lambda(h=1)}{dh} h_1 \tag{6}$$

$$Y_i = \bar{Y}_i + \epsilon Y_{1i} + o(\epsilon) \qquad \sum_{i=0}^{N} Y_{1i} = 0 \tag{7}$$

Substitution of these expansions in the nondimensional hydrodynamic conservation equations for an ideal gas yields the Boussinesq equations:

$$\nabla p_e = \hat{x} \qquad \nabla\cdot\mathbf{q}_1 = 0$$
$$\rho_1 = -h_1 - \left[\sum_{i=1}^{N}\frac{Y_{1i}}{m_i}\Big/\sum_{i=1}^{N}\frac{\bar{Y}_i}{m_i}\right] \tag{8}$$

$$\epsilon G[\nabla(q_1^2/2) + (\nabla\, x\, \mathbf{q}_1)\, x\, \mathbf{q}_1] = -\nabla p_1 + \rho_1\hat{x} - \nabla x(\nabla\, x\, \mathbf{q}_1) \tag{9}$$

$$\epsilon G\mathbf{q}_1\cdot\nabla h_1 = (1/Pr)\nabla^2 h_1 \tag{10}$$

$$\epsilon G\mathbf{q}_1\cdot\nabla Y_{1i} = (1/Sc_i)\nabla^2 Y_{1i} \qquad \imath = 1, 2, \ldots, \mathrm{N} \tag{11}$$

subject to

$$\frac{\partial Y_{1i}(\mathbf{z}_1)}{\partial n_1} = \frac{\partial Y_{1i}(\mathbf{z}_2)}{\partial n_2} = \mathbf{q}_1(\mathbf{z}_1) = \mathbf{q}_1(\mathbf{z}_2) = h_1(\mathbf{z}_1) = h_1(\mathbf{z}_2) - 1 = 0 \tag{12}$$

Received August 2, 1967; revision received January 22, 1968. These remarks are drawn from informal notes prepared for the Environmental Science Services Agency under Contract Cwb-11310 in August 1966.

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provided $M^2 \ll \min(\epsilon G, 1)$. The parameter ϵG is the Grashof number. Clearly these results are independent of the particular choice for λ_e, as well as the particular dependence of either viscosity coefficient on enthalpy.

The form adopted by Lowell and Adams for the buoyant force, once the hydrostatic pressure is removed, is $[(\rho_\infty - \rho) \div \rho]g = (\rho_\infty/\rho - 1)g$. For ρ limited to a small perturbation about its ambient value, this form, under proper definition of constants, can be expanded into the conventional linear form given previously. Lowell and Adams do not mention any limitations on their model; therefore, their unusual form for the density variation may be an attempt to modify the Bossinesq approximation to a form suitable for larger density variations. However, Eshghy and Morrison,[2] for the special case of an isothermal semi-infinite vertical flat plate, have obviated the need for any ad hoc approximate forms by discovering a kind of free-convection correlation of the compressible and incompressible boundary-layer equations. Eshghy and Morrison's results agree with the present formulation in that a vanishingly small temperature difference is sufficient for the Boussinesq equations to be valid. In summary, for small temperature differences, the Lowell-Adams form is equivalent to a conventional Boussinesq approximation; for larger temperature differences, other procedures are on a firmer basis. It is noted that, prior to the Eshghy and Morrison work, Sparrow and Gregg[10] had attempted to consider compressibility effects in the vertical-flat-plate free-convection problem, but their analysis is less complete.

The Pohlhausen similarity succeeds for a wall (say of some pure nongaseous substance, designated species 1) uniformly near its sublimation (or vaporization) temperature in the presence of hot ambient gas (the inverse problem of condensation at the wall of a species from a cold supersaturated gaseous state can be similarly treated).[11–13] Here the quasi-steady vaporization case is examined and the rate of mass transfer is sought. In view of the aforementioned derivation, it seems reasonable to adopt the following (dimensional) governing equations for the case $1 \gg \epsilon > 0$ where now $\epsilon = (T_\infty - T_w)/T_\infty$ (see Fig. 1):

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \tag{13}$$

$$L_{Di}\tilde{Y}_i = L_K\tilde{T} = L_\nu u + \frac{\partial \tilde{p}}{\partial x} + g\left(\beta\tilde{T} + \sum_{i=1}^{N} \delta_i \tilde{Y}_i\right) = 0 \tag{14}$$

$$\partial\bar{p}/\partial y = 0 \tag{15}$$

$$\bar{p} = \rho R T \sum_{i=1}^{N} \frac{Y_i}{m_i} \tag{16}$$

where

$$\tilde{p} = \bar{p} - p_e(x) \qquad \tilde{Y}_i = Y_i - Y_{i\infty} \qquad \tilde{T} = T - T_\infty \tag{17}$$

and

$$L_Q \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - Q\frac{\partial^2}{\partial y^2} \tag{18}$$

$$\beta = -\frac{1}{\rho_\infty}\left(\frac{\partial \rho}{\partial T}\right)_\infty = \frac{1}{T_\infty} \qquad \delta_i = \frac{-1}{\rho_\infty}\left(\frac{\partial\rho}{\partial Y_i}\right)_\infty = \frac{1}{m_i}\left(\sum_{j=1}^{N}\frac{Y_{j\infty}}{m_j}\right)^{-1} \tag{19}$$

subject to

$$y \to \infty: \quad \tilde{p} \to 0, \quad u \to 0, \quad \tilde{Y}_i \to 0, \quad T \to 0 \tag{20}$$

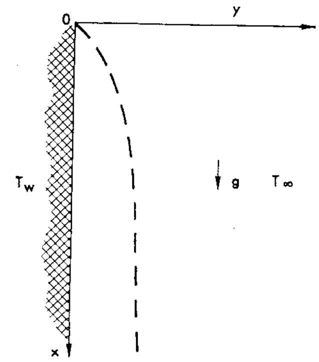

Fig. 1 Geometry of the vertical flat-plate boundary layer for the case of an adiabatically vaporizing wall.

$y = 0$:

$$u = 0 \tag{21}$$

$$\frac{\partial \tilde{T}}{\partial y} = \chi v \qquad \chi = \frac{\rho_\infty \bar{\lambda}}{k} \tag{22}$$

$$v\tilde{Y}_j = D_j \frac{\partial \tilde{Y}_j}{\partial y} - vY_{j\infty} \qquad j = 2, 3, \ldots, N \tag{23}$$

$$v\tilde{Y}_1 = D_1\frac{\partial \tilde{Y}_1}{\partial y} + v(1 - Y_{1\infty}) \tag{24}$$

$$\tilde{Y}_1 = -Y_{1\infty} + \exp\left[\Phi\left(\frac{1}{T_B} - \frac{1}{T_\infty + \tilde{T}}\right)\right] \qquad \Phi = \frac{\bar{\lambda} m_1}{R} \tag{25}$$

Equation (22) is the adiabatic vaporization condition; Eq. (25), an approximate form of the Clausius-Clapeyron equation.[9] The quantity $v(0)$ gives the mass transfer rate. Obviously $\tilde{p} = 0$ everywhere in the lowest-order boundary-layer problem stated previously.

Pohlhausen similarity analysis leads to

$$u = 4\nu C^2 x^{1/2}\phi'(\xi) \qquad v = \nu C x^{-1/4}(\xi\phi' - 3\phi) \tag{26}$$

$$\tilde{Y}_i = \tilde{\tilde{Y}}_i(\xi) \qquad \tilde{T} = \theta(\xi) \qquad \xi = Cy/x^{1/4} \qquad C = (\beta g/4\nu^2)^{1/4} \tag{27}$$

The two-point boundary-value problem (13–25) becomes

$$\theta'' + 3Pr\phi\theta' = 0 \qquad \tilde{\tilde{Y}}_i'' + 3Sc_i\phi\tilde{\tilde{Y}}_i' = 0 \qquad i = 1, 2, \ldots, N \tag{28}$$

$$\phi''' + 3\phi\phi'' - 2\phi'^2 - \theta - \beta^{-1}\sum_{i=1}^{N}\delta_i\tilde{\tilde{Y}}_i = 0 \tag{29}$$

$$\xi \to \infty: \quad \theta \to 0, \quad \tilde{\tilde{Y}}_i \to 0, \quad \phi' \to 0 \tag{30}$$

$\xi = 0$:

$$\phi' = 0 \qquad \theta' = -3Pr\chi_1\phi \qquad \chi_1 = \frac{\bar{\lambda}}{c_p} \tag{31}$$

$$\tilde{\tilde{Y}}_i' = -3Sc_i\phi(\tilde{\tilde{Y}}_i + Y_{i\infty}) \qquad i = 2, 3, \ldots, N \tag{32}$$

$$\tilde{\tilde{Y}}_i' = -3Sc_1\phi(\tilde{\tilde{Y}}_1 - 1 + Y_{i\infty}) \tag{33}$$

$$\tilde{\tilde{Y}}_1 = -Y_{1\infty} + \exp\left[\Phi\left(\frac{1}{T_B} - \frac{1}{T_\infty + \theta}\right)\right] \tag{34}$$

Finally, it is remarked that the perturbation used in Eq. (14) is often adopted in conjunction with equations of state other than that which is appropriate for a mixture of ideal gases, Eq. (16).

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Reply by Author to F. M. White*

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THE author appreciates the interest of F. M. White in his paper. It is unfortunate that the reference to Spalding's earlier work has been overlooked. Spalding's formula represents the constant shear version of the more generalized form derived in Ref. 1. It should be noted that, although this formula appears to correlate adequately with the velocity distribution in the whole pipe, it yields an incorrect shear distribution in the outer regions when it is applied consistently with the eddy viscosity proposed by Spalding. The seemingly adequate representation of velocities is due to the fact that up to 70 or 80% of the velocity distribution does lie in a nearly constant shear region, whereas at the tail of the distribution almost any smooth monotonic function will close the gap reasonably well. No attempt has been made in the paper to adjust the numerical constants to fit experimental data; rather, they were taken from the literature, i.e., $k_1 = 0.4$ and $k_2' = 5.1$, which in turn yields $k_2 = 7.7$.

Received October 19, 1967.

* The Technical Comment by F. M. White, which was published in Vol. 6, No. 4, p. 767, April 1968, should have appeared with this Reply.

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It has been stated in the report and may be reiterated here that, even with the correction for the shear distribution, the application of the derived law must remain within the wall region. In regions where the convective terms become substantially larger than the leading term in the expression for the shear distribution the flow no longer depends on local conditions only and, therefore, it is outside the range of applicability of the proposed law.

The generalized form of the law of the wall,[1] when applied to such diversified cases as the injection problem and the investigation of the growth of the viscous sublayer, predicts results that are not refuted by experiments; this fact provides sufficient reason to believe that the results are not fortuitous as contended by the commentator.

Reference

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Comment on "Pressure Distributions on Sharp Cones in Rarefied Hypersonic Flow"

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IN a recent Note, McCroskey[1] has given an interesting correlation of a large quantity of surface pressure data from various sources taken on sharp cones at hypersonic semi-rarefied conditions. McCroskey also compares the data with the weak interaction theory of Burke and Dowling[2] and the strong interaction theory of Stewartson.[3] In the region where it is valid, the weak interaction theory agrees reasonably with the data. The strong interaction theory, however, agrees poorly, particularly in the region where one would expect it to agree best. The purpose of the present Comment is to show that this disagreement is primarily due to an algebraic mistake in the analysis of the heat-transfer case in Ref. 3. This error was noted and corrected by the present author in Ref. 4.

The mistake in Ref. 3 occurs in passing from Eq. (4.21)† to Eqs. (4.23) and (4.24) [although there is apparently a misprint in Eq. (4.21)]. Equation (4.23) should read

$$[4/(1 + S_w)]\epsilon_0^2 = C$$

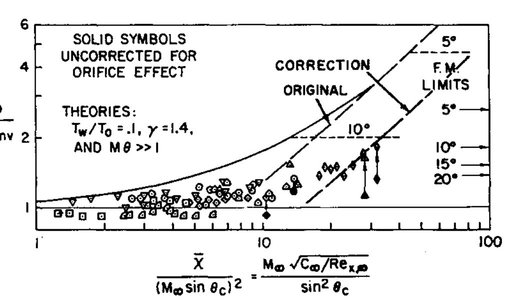

Fig. 1 Surface pressure distributions on highly cooled sharp cones.

Received November 29, 1967.

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† Numbered equations referred to herein are equations of Ref. 3; the notation is also that of Ref. 3.